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QN.→ Define inner-Product and inner Product Space.

(Defn) Inner Product Space:- Let E be a linear space over a field K. A map of $E \times E$ into K which associates to each element (x, y) of $E \times E$ a unique scalar denoted by (x/y) and called the inner product or the scalar product of x & y is called the inner Product map if the following condition also satisfied:-

(I) $(x/x) \geq 0$ for every $x \in E$;

(II) $(x/x) = 0$ iff $x = 0$

(III) $(x/y) = \overline{(y/x)}$ for all $x, y \in E$. $\overline{(y/x)}$ denotes the Complex Conjugate of (y/x).

(IV) $(\alpha x + \beta y/z) = \alpha(x/z) + \beta(y/z)$ for all $x, y, z \in E$ and $\alpha, \beta \in K$.

A linear space E together with an inner Product map on it is called an inner Product space.

Remark, By (III) $\overline{(x/x)} = (x/x)$ ∴ (x/x) is a real no. for every $x \in E$.

(I) If $K = R$, then (III) reduces to ~~(IV)~~ for every $x, y, z \in E$ and every $\alpha, \beta \in K$,

$$(x/\alpha y + \beta z) = \bar{\alpha}(x/y) + \bar{\beta}(x/z).$$

Proof:- $(x/\alpha y + \beta z) = \overline{(\alpha y + \beta z/x)}$

$$= \overline{\alpha(y/x) + \beta(z/x)} = \bar{\alpha} \cdot \overline{(y/x)} + \bar{\beta}\,\overline{(z/x)}$$

$$= \bar{\alpha}(x/y) + \bar{\beta}(x/z).$$

An Inner Product Space is called a real or Complex inner Product space a as the field K is the field of real or Complex nos.

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Inner Product norm:- Let E be an inner Product Space for every $x \in E$, we define $\|x\| = +\sqrt{(x/x)}$. We verify that this actually defines a norm on E.

93 QNo- Show that an inner Product Space E is a normed linear space w.r.t. the norm defined by $\|x\| = +(x,x)^{1/2}$ $\forall x \in E$

Verification:- Clearly, $\|x\| \geq 0$ for every $x \in E$.
Also, $\|x\| = 0$ iff $+\sqrt{x/x} = 0$ iff $(x/x) = 0$ iff $x = 0$

Also for every $\alpha \in K$ and every $x \in E$,

$$\|\alpha x\|^2 = (\alpha x/\alpha x) = \alpha(x/\alpha x) = \alpha\bar{\alpha}(x/x) = |\alpha|^2 \cdot \|x\|^2$$

$$\therefore \|\alpha x\| = |\alpha| \cdot \|x\|$$

In order to verify the triangle inequality, we shall first Prove the following Cauchy Schwarz inequality

For any $x, y \in E$, $|(x/y)| \leq \|x\| \cdot \|y\|$

When, $y = 0$ the equality holds, So let $y \neq 0$.
For any Scalar λ, we have

$$0 \leq \|x - \lambda y\|^2 = (x - \lambda y / x - \lambda y)$$
$$= (x/x) - \bar{\lambda}(x/y) - \lambda(y/x) + \lambda\bar{\lambda}(y/y).$$
$$= \|x\|^2 + |\lambda|^2 \|y\|^2 - \bar{\lambda}(x/y) - \lambda\overline{(x/y)}$$

Taking $\lambda = \frac{(x/y)}{\|y\|^2}$, $0 \leq \|x\|^2 + \frac{|(x/y)|^2}{\|y\|^4} \cdot \|y\|^2$

$$- \frac{\overline{(x/y)}(x/y)}{\|y\|^2} - \frac{(x/y)\overline{(x/y)}}{\|y\|^2}.$$

or, $0 \leq \|x\|^2 + \frac{|(x/y)|^2}{\|y\|^2} - \frac{2|(x/y)|^2}{\|y\|^2}$

or, $0 \leq \|x\|^2 - \frac{|(x/y)|^2}{\|y\|^2}$

$\therefore \frac{|(x/y)|^2}{\|y\|^2} \leq \|x\|^2$

or, $|(x/y)|^2 = \|x\|^2 \cdot \|y\|^2$

or, $|(x/y)| \leq \|x\| \cdot \|y\|$.

Now, for any $x, y \in E$, $0 \leq \|x+y\|^2 = (x+y / x+y)$

$= (x/x) + (x/y) + (y/x) + (y/y)$

$= \|x\|^2 + \|y\|^2 + (x/y) + \overline{(x/y)}$ ~~=~~

$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x/y)$.

Now, by Cauchy-Schwarz inequality,

$|x/y| \leq \|x\| \cdot \|y\|$

$\therefore 2\,\mathrm{Re}(x/y) \leq 2|(x/y)| \leq 2\|x\| \cdot \|y\|$

$\therefore 0 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$

i.e. $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\therefore \|x+y\| \leq \|x\| + \|y\|$.

Hence, every inner product space is a n.l.s. w.r.t. the inner product norm defined by,

$\|x\| = +\sqrt{(x/x)}$ for every $x \in E$.

i.e. $\|x\| = +(x/x)^{1/2}$ $\forall$ $x \in E$.

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Q.No → Give an example of inner-Product Space.

Example: - Let X denote the inner Product Space of all real valued Continuous functions on $[0, 2\pi]$ with inner Product defined by,

$$(x/y) = \int_0^{2\pi} x(t)\, y(t)\, dt,$$

If $u_n(t) = \cos nt$, then by integration, we get

$$(u_m/u_n) = \int_0^{2\pi} \cos mt \cos nt\, dt = 0 \text{ if } m \neq n.$$
$$= \pi \text{ if } m = n = 1, 2, ---$$
$$= 2\pi \text{ if } m = n = 0,$$

Therefore, if $e_0(t) = \frac{1}{\sqrt{2\pi}}$ & $e_n(t) = \frac{\cos nt}{\sqrt{\pi}}$, $n = 1, 2, ---$

then we see that $\{e_n\}$ is an orthonormal Sequence in X. By taking $v_n(t) = \sin nt$, we may Construct Similarly another Set of orthonormal sequence in X.